

Brane Holes

Valeri P. Frolov*

Theoretical Physics Institute, University of Alberta, Edmonton, AB, Canada, T6G 2G7

Shinji Mukohyama†

*Institute for the Physics and Mathematics of the Universe (IPMU)
The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8582, Japan
(Dated: February 21, 2011)*

The aim of this paper is to demonstrate that in models with large extra dimensions under special conditions one can extract information from the interior of $4D$ black holes. For this purpose we study an induced geometry on a test brane in the background of a higher dimensional static black string or a black brane. We show that at the intersection surface of the test brane and the bulk black string/brane the induced metric has an event horizon, so that the test brane contains a black hole. We call it a *brane hole*. When the test brane moves with a constant velocity V with respect to the bulk black object it also has a brane hole, but its gravitational radius r_e is greater than the size of the bulk black string/brane r_0 by the factor $(1 - V^2)^{-1}$. We show that bulk ‘photon’ emitted in the region between r_0 and r_e can meet the test brane again at a point outside r_e . From the point of view of observers on the test brane the events of emission and capture of the bulk ‘photon’ are connected by a spacelike curve in the induced geometry. This shows an example in which extra dimensions can be used to extract information from the interior of a lower dimensional black object. Instead of the bulk black string/brane, one can also consider a bulk geometry without horizon. We show that nevertheless the induced geometry on the moving test brane can include a brane hole. In such a case the extra dimensions can be used to extract information from the complete region of the brane hole interior. We discuss thermodynamic properties of brane holes and interesting questions which arise when such an extra dimensional channel for the information mining exists.

PACS numbers: 04.70.Bw, 04.70.-s, 04.25.-g

Alberta-Thy-14-10, IPMU-10-0219

I. INTRODUCTION

Models with large extra dimensions have been ‘popular’ and intensively discussed since 1998 [1–3]. In these models our four dimensional spacetime Σ is considered as a brane embedded in a higher dimensional bulk space. The main purpose of the present paper is to demonstrate that in such models under special conditions extra dimensions can be used to extract information from the interior of a four dimensional black hole. This happens when the four dimensional surface Σ representing our world is not geodesic. Consider two points on Σ and suppose that they can be connected by a geodesic in the bulk spacetime. Let us assume that there also exists a curve on Σ which connects the two points and which is geodesic in the induced geometry. In this case the geodesic distance between the two points along the bulk geodesic is in general different from the geodesic distance in the induced geometry. Under special conditions, for two points on Σ separated by a spacelike induced interval, there may exist causal curves connecting them through the bulk spacetime. An observer on the brane Σ would describe this situation by saying that the extra dimensions provide one with a channel of information exchange with an effective super-luminal velocity.

A simple model demonstrating such a possibility was considered in [4]. In the paper a stationary cosmic string in the Kerr geometry was studied. The $2D$ induced metric on the string worldsheet has a horizon at its intersection with the ergosurface of the bulk geometry. It was shown that the $2D$ black hole produces the Hawking radiation of the string transverse degrees of freedom [5], so that the cosmic string can be used to mine energy from the bulk black hole [6].

In this paper we study more ‘realistic’ case where a brane representing our $4D$ world is embedded in a higher dimensional bulk spacetime. We neglect effects connected with the thickness of the brane. We also neglect the gravitational field generated by the brane and use the test brane approximation. To simplify the presentation in the most of the paper we discuss the case in which the bulk space has five dimensions. Generalization to other dimensions of the brane and the bulk spacetime is straightforward and is briefly discussed at the end of the paper. We discuss two models. In the first model the test brane is moving with a constant velocity (as measured by a distant observer) in the bulk space with a black string. We show that if r_0 is the gravitational radius of the black string, the induced geometry has a $4D$ black hole with the larger radius $r_e = (1 - V^2)^{-1}r_0$. We demonstrate that the test brane embedding is not geodesic, and extra dimensions can be used to extract information from the region between r_0 and r_e of the induced brane hole. In the second model we replace a bulk black string by a spacetime with a static massive thin shell of mass M and

*Electronic address: vfrolov@ualberta.ca

†Electronic address: shinji.mukohyama@ipmu.jp

radius r_s . We shall see later that the gravitational radius of the shell is $r_0 = 2M - M^2/r_s \leq r_s$ and that the bulk spacetime is regular and does not contain a bulk black object. Nonetheless, if r_e is greater than r_s then the induced metric has a brane hole. We demonstrate that in this model the complete brane hole interior is ‘visible’ through extra dimensions.

The existence of the extra dimensional ‘window’ for observing the brane hole interior raises a number of interesting questions. Some of them will be briefly discussed in the paper.

The rest of this paper is organized as follows. In Sec. II two five-dimensional geometries describing a black string and a dark shell at rest are presented. In Sec. III these static bulk geometries are boosted so that they describe a moving string and a moving dark shell, respectively. Sec. IV describes a test brane in the background of the boosted black string and shows that the induced metric on the test brane contains a brane hole. Sec. V shows some thermodynamic properties of brane holes. In Sec. VI it is shown that information can be transmitted via the bulk space from the inside of a brane hole to the outside. Sec. VII describes a test brane in the boosted dark shell geometry and shows that the induced metric can still have a brane hole. Sec. VIII is devoted to summary of the paper and discussions.

II. BLACK STRING AND DARK SHELL

In this paper we study properties of a moving test brane in an external gravitational field. For the latter we use two models: (i) a bulk black brane, and (ii) a dark shell model. In this subsection we discuss the first model. For simplicity we assume the number of dimensions of the bulk spacetime is equal to five, so the bulk black brane in fact is a black string. We also assume that the test brane has a codimension one, so that the induced geometry on it is four dimensional. In the both cases one can compactify the metric in the fifth flat dimension, so that the corresponding z -coordinate becomes periodic with some period L . Such a compactification as well as generalizations to higher number of dimensions, which is quite straightforward, are briefly discussed in Appendices.

A. Bulk black string

The 5-dimensional metric of a black string is ($a, b = 0, 1, \dots, 4$)

$$dS^2 = g_{ab} dy^a dy^b = -\Phi d\tilde{t}^2 + \frac{dr^2}{\Phi} + r^2 d\omega^2 + d\tilde{z}^2, \quad (1)$$

$$\Phi = 1 - \varphi, \quad \varphi = 1/r, \quad (2)$$

where $d\omega^2$ is a line element of a unit sphere S^2 ,

$$d\omega^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (3)$$

This metric is a direct sum of the Schwarzschild metric and a line. We write the metric in the dimensionless form, that is put the gravitational radius $2M$ equal to one. To restore proper dimensionality it is sufficient to multiply coordinates \tilde{t} , \tilde{z} and r by the factor $(2M)^{-1}$ and rescale the metric as $dS^2 \rightarrow (2M)^2 dS^2$.

Denote

$$\tilde{\eta}^a = \xi_{(\tilde{t})}^a + v \xi_{(\tilde{z})}^a, \quad (4)$$

where $\xi_{(\tilde{t})}^a \partial_a = \partial_{\tilde{t}}$ and $\xi_{(\tilde{z})}^a \partial_a = \partial_{\tilde{z}}$ are two commuting Killing vectors generating transformations along \tilde{t} and \tilde{z} coordinates, respectively. The norm of this vector is

$$\tilde{\eta}^2 = -\Phi + v^2. \quad (5)$$

The vector $\tilde{\eta}^a$ is timelike for $v^2 < \Phi$ and becomes null when $v = \pm\sqrt{\Phi}$. This means that there exist a Killing observer with the velocity $u^a \sim \tilde{\eta}^a$ at the radius r only if its relative velocity v with respect to a rest frame is in the interval $v \in (-\sqrt{\Phi}, \sqrt{\Phi})$. Near the horizon where Φ vanishes, this interval shrinks to zero. One can say that the particle motion in z -direction is frozen.

B. Dark shell

In our second model we assume that outside some radius $r_s = 1 + \varepsilon$ with small positive ε the metric coincides with (1) and inside this radius it is flat

$$dS_-^2 = -\frac{\varepsilon}{1+\varepsilon} d\tilde{t}^2 + dr^2 + r^2 d\omega^2 + d\tilde{z}^2. \quad (6)$$

In what follows we denote the external metric as g_{+ab} . We choose the form of the metric so that all the metric coefficients, except g_{rr} are continues at the junction surface Σ_ε . The jump of the coefficient g_{rr} implies the jump of an extrinsic curvature. Thus the spacetime contains a thin massive shell.

Using Israel’s method [7] it is possible to find the shell parameters: the surface energy density σ and components of the pressure $P_{z,\theta}$.

$$8\pi\sigma = \frac{2}{1+\varepsilon} \left(1 - \sqrt{\frac{\varepsilon}{1+\varepsilon}} \right),$$

$$8\pi P_z + 8\pi\sigma = 8\pi P_\theta + 4\pi\sigma = \frac{1}{2\varepsilon^{1/2}(1+\varepsilon)^{3/2}}. \quad (7)$$

Note that the mass of the shell is

$$M = 4\pi\sigma r_s^2 = r_s \left(1 - \sqrt{1 - \frac{r_0}{r_s}} \right), \quad (8)$$

where $r_0 = 1$ and $r_s = (1+\varepsilon)$ are the horizon radius of the bulk black string and the radius of the shell, respectively. Thus we obtain

$$r_0 = 2M - \frac{M^2}{r_s}. \quad (9)$$

The red-shift factor on the outer surface of the shell is $\sqrt{\varepsilon/(1+\varepsilon)}$. For small ε this factor is small. This explains an adopted in this paper terminology a *dark shell*.

III. BOOSTED METRICS

Our purpose is to consider a moving test brane in the described geometries. We assume that the test brane is asymptotically flat and moved with a constant velocity V in z -direction with respect to either a black string or a dark shell. This problem is equivalent to the case when the test brane is asymptotically flat and is at rest, while the black string and the dark shell moves with the constant velocity $-V$ in z -direction. The gravitational field of such objects can be easily obtained by a boost transformation.

A. Boosted black string

In order to obtain a metric of a boosted black string, let us make the following transformation

$$\tilde{t} = ct + sz, \quad \tilde{z} = st + cz, \quad (10)$$

$$c = \cosh \alpha, \quad s = \sinh \alpha. \quad (11)$$

Here α is a boost parameter. This transformation generates the motion in z -direction with the velocity $V = s/c$, and c is the corresponding Lorentz γ -factor. Applying this transformation to the black string metric one obtains

$$dS^2 = -dt^2 + dz^2 + \varphi(ct + sz)^2 + \frac{dr^2}{1-\varphi} + r^2 d\omega^2. \quad (12)$$

To compactify the boosted black string metric one need to assume that z -coordinate is periodic with the period L . Notice that operations of the boost and compactification do not commute. Thus the geometry of compactified spacetime of the moving black string is different from the unboosted one, where the periodicity is imposed in the \tilde{z} coordinate.

The metric (12) has a coordinate singularity at the black string horizon $r = 1$. It can be removed by the following coordinate transformation

$$t = v - cr_*, \quad z = y + s \ln \Phi, \quad r_* = \int \frac{dr}{\Phi} = r + \ln(r-1). \quad (13)$$

The metric in these coordinates is

$$\begin{aligned} dS^2 = & -(1-c^2\varphi)dv^2 + (1+s^2\varphi)dy^2 \\ & -s^2(1+\varphi)(1-c^2\varphi-s^2\varphi^2)dr^2 \\ & +2cs\varphi dv dy - 2s\varphi(c^2+s^2\varphi)dy dr \\ & +2c(1-s^2\varphi-s^2\varphi^2)dr dv. \end{aligned} \quad (14)$$

The new coordinates (v, r, y, θ, ϕ) cover both exterior and interior of the black string. These coordinates are similar to the advance time for the Eddington-Finkelstein coordinates in the Schwarzschild spacetime. By changing the signs in (13) one can define u coordinate which is an analogue of the retarded time.

B. Ergoregion

An important new feature of the spacetime (12) is an existence of the *ergoregion*. The metric (12) has two commuting Killing vectors $\xi_{(t)}^a \partial_a = \partial_t$ and $\xi_{(z)}^a \partial_a = \partial_z$. One has

$$(\xi_{(t)})^2 = g_{tt} = -(1-c^2\varphi). \quad (15)$$

Hence the vector $\xi_{(t)}^a$ is timelike at the spatial infinity, it becomes null at $r = r_e = c^2$, and it is spacelike inside this surface. The infinite red-shift surface r_e is located outside the boosted bulk brane horizon $r = 1$. We call the spacetime region between $r = r_e$ and 1 an *ergoregion*, and its external boundary, $r = r_e$ an *ergosurface*.

The ergoregion has an important characteristic property: causal propagation inside it is always a motion with the decrease of the coordinate z . To demonstrate this let us consider a linear combination of the Killing vectors

$$\eta = \xi_{(t)}^a + v\xi_{(z)}^a. \quad (16)$$

Its square is

$$\eta^2 = -(1-c^2\varphi) + 2vsc\varphi + v^2(1+s^2\varphi). \quad (17)$$

The velocity of an observer who is at rest with respect to the infinity is proportional to $\xi_{(t)}^a$. An observer with the velocity directed along the vector η is moving with respect to the rest frame with the speed v . A condition $\eta^2 = 0$ determines an intersection of a local null cone with $(t-z)$ -plane. Solving this equation we find

$$v_{\pm} = \frac{-sc\varphi \pm \sqrt{1-\varphi}}{1+s^2\varphi}. \quad (18)$$

Causal motion with constant r is impossible inside the surface $r = 1$. This region is the bulk black string interior. Notice that the sign of v_- is always negative. v_+ vanishes when

$$sc\varphi = \sqrt{1-\varphi}. \quad (19)$$

This equation has two roots

$$\varphi = c^{-2} \text{ and } \varphi = -s^{-2}. \quad (20)$$

The second root gives $r = -s^2$ and hence it is unphysical. The first root determines $r = r_e = c^2$. This is an equation of the ergosurface. Inside the ergosurface both v_+ and v_- are negative. Hence in the ergoregion particles and light always propagate with decrease of the coordinate z . In

their motion the radius r can become smaller or larger. Hence a particle and light can leave the ergoregion. Inside 1 the motion always reduces r , so that the horizon 1 is a surface of ‘no return’. These properties are similar to the properties of stationary rotating black holes.

C. Boosted dark shell

The metric (12) describes also the external metric for the boosted dark shell. The inner metric can be obtained easily by applying the boost transformation (10) to (6) and it is of the form

$$dS_-^2 = -\frac{\varepsilon - s^2}{1 + \varepsilon} dt^2 + \frac{2sc}{1 + \varepsilon} dt dz + \frac{c^2 + \varepsilon}{1 + \varepsilon} dz^2 + dr^2 + r^2 d\omega^2. \quad (21)$$

IV. TEST BRANE IN THE BOOSTED BLACK STRING SPACETIME

A. Test brane equation

We refer to the spacetime with metric (12) as a *bulk space*. We assume now that in the bulk space there exists a $4D$ brane which represents our ‘physical’ spacetime. Such a brane is a $4D$ submanifold Σ embedded in the $5D$ bulk manifold. We assume that the brane ‘respects’ the symmetries of the bulk space, that is it is static and spherically symmetric and choose the equation for the embedding in the form

$$F = z - Z(r) = 0. \quad (22)$$

The induced metric on the test brane is $(\mu, \nu = 0, 1, 2, 3)$

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu = -(1 - c^2\varphi) dt^2 + 2scZ'\varphi dt dr + \left[(1 + s^2\varphi)Z'^2 + \frac{1}{1 - \varphi} \right] dr^2 + r^2 d\omega^2. \quad (23)$$

Here $(\dots)' = d(\dots)/dr$. The induced metric can be diagonalized by the following coordinate redefinition

$$T = t - sc \int \frac{dr\varphi Z'}{1 - c^2\varphi}. \quad (24)$$

In the new coordinates the metric (23) is

$$ds^2 = -(1 - c^2\varphi) dT^2 + \left[Z'^2 \frac{1 - \varphi}{1 - c^2\varphi} + \frac{1}{1 - \varphi} \right] dr^2 + r^2 d\omega^2. \quad (25)$$

Simple calculations give the the following expression for the determinant g of the induced metric

$$g = -r^4 \sin^2 \theta \frac{Z'^2(1 - \varphi)^2 + 1 - c^2\varphi}{1 - \varphi}. \quad (26)$$

We choose the brane action in the form

$$W = \int dT \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \int dr \sqrt{-g} \\ = 4\pi \Delta T \int dr L \quad (27)$$

$$L = r^2 \sqrt{\frac{Z'^2(1 - \varphi)^2 + 1 - c^2\varphi}{1 - \varphi}}. \quad (28)$$

Here ΔT is an interval of the time T . An extremum of the action W determines a minimal surface Σ , which is the world-volume of the brane. By varying (27) one obtains the following equation

$$\frac{d}{dr} \left[\frac{r^2(1 - \varphi)^{3/2} Z'}{\sqrt{Z'^2(1 - \varphi)^2 + 1 - c^2\varphi}} \right] = 0. \quad (29)$$

This equation implies that the expression in the square brackets is a constant. We denote this constant by B , then one has

$$Z' = \frac{B}{1 - \varphi} \sqrt{\frac{1 - c^2\varphi}{r^4(1 - \varphi) - B^2}}. \quad (30)$$

B. Induced geometry

Substitution of (30) in (25) gives

$$ds^2 = -(1 - c^2\varphi) dT^2 + \left[\frac{r^4}{r^4(1 - \varphi) - B^2} \right] dr^2 + r^2 d\omega^2. \quad (31)$$

The equation (30) contains an arbitrary parameter B . It can be fixed by imposing the condition that the induced metric is regular at the surface r_e where it crosses the ergosurface.

The Ricci scalar for the induced metric is

$$R = \frac{c^2 s^2 r^4 - 12B^2 r^2 + 20B^2 c^2 r - 9c^4 B^2}{2r^6(r - c^2)^2}. \quad (32)$$

The Ricci scalar of the induced metric is regular for $r > c^2$. In a general case it is divergent at

$$r = r_e = c^2. \quad (33)$$

The regularity condition at this point singles out a special value of the integration constant B

$$B = \pm sc^3. \quad (34)$$

For this value the Ricci scalar takes the form

$$R = \frac{s^2 c^2 (r - r_+)(r - r_-)}{2r^6}, \quad r_{\pm} = (-1 \pm \sqrt{10})c^2. \quad (35)$$

For this choice of B the induced metric is regular at $r = r_e$. It takes the form

$$ds^2 = -\left(1 - \frac{c^2}{r}\right)dT^2 + \frac{r^4}{(r - c^2)U}dr^2 + r^2d\omega, \quad (36)$$

where

$$U = r^3 + s^2(r^2 + c^2r + c^4). \quad (37)$$

This geometry represents a 4-dimensional black hole induced on the brane. The expression (33) gives the size of its horizon and, for $V \neq 0$, it is greater than that of the bulk black string. It is easy to understand the physical reason for this. For $V \neq 0$, i.e. for a moving brane, a null vector tangent to the brane world-volume always has a non-vanishing \tilde{z} -component from the bulk point of view. As a result, its r -component is smaller than the speed of light. This means that a null geodesic on the brane is easier to be trapped by gravity of the black string than a radial null geodesic in the bulk. In particular, there exist outward null geodesics on the brane which start from points slightly outside the black string horizon and are still trapped by gravity of the black string. This explains the physical reason why the horizon defined by the brane-induced geometry is greater than the black string horizon in the bulk. We shall give an alternative explanation for this fact at the end of this section.

The proper distance to the brane-hole horizon is finite. At $r = c^2$ one has $U = c^4(4c^2 - 3)$. The proper distance from $r = c^2$ to a nearby point r is

$$\rho \sim \frac{2c^2}{\sqrt{4c^2 - 3}}\sqrt{r - c^2}. \quad (38)$$

For the metric (36), not only the Ricci scalar but also other curvature invariants remain finite. (See, for example the explicit form of the Ricci tensor presented in the Appendix B.) This regularity follows from the following observation. Denote

$$r = c^2 + \frac{4c^2 - 3}{4c^4}\rho^2, \quad (39)$$

then near $r \approx c^2$ one has

$$ds^2 \approx -\kappa^2\rho^2dT^2 + d\rho^2 + c^2d\omega^2. \quad (40)$$

This metric has the Rindler form in the (T, ρ) sector. At $\rho = 0$ there exists a horizon. The coordinate ρ has the meaning of the proper distance from the horizon, and

$$\kappa = \frac{\sqrt{4c^2 - 3}}{2c^3} \quad (41)$$

is the surface gravity of the horizon.

For the regular test brane one has

$$Z' = \pm \frac{sc^3\sqrt{r}}{(r - 1)\sqrt{U}}. \quad (42)$$

Note that this expression is well-defined except at $r = 1$, while the regime of validity of (30) is more restrictive for other values of B . At large distance $Z' \sim \pm sc^3/r^{7/2}$. Hence

$$Z \sim Z_0 \mp \frac{2}{5} \frac{sc^3}{r^{5/2}}. \quad (43)$$

Function $Z(r)$ rather fast reaches its asymptotic value Z_0 . At the black string horizon $U = U(1) = c^6$ and one has

$$Z' \sim \pm \frac{s}{r - 1}, \quad (44)$$

so that

$$Z \sim \pm s \ln(r - 1). \quad (45)$$

The embedding function $Z(r)$ is singular at the black string horizon $r = 1$. This is an consequence of the coordinate singularity of the metric (12) at this point. This singularity can be removed by the coordinate transformations (13). In the new coordinates (v, r, y, θ, ϕ) the brane equation is $y = Y(r)$

$$Y' = \frac{sc^3\sqrt{r}}{(r - 1)\sqrt{U}} - \frac{s}{r(r - 1)}. \quad (46)$$

We chose the $+$ sign in (42). For the opposite sign one needs to use the retarded time u . Relation (44) shows that $Y(r)$ is regular at $r = 1$. Figure 1 shows some examples of the function $Y(r)$.

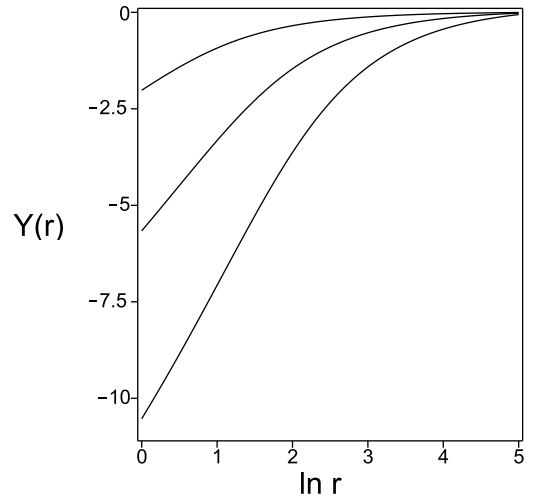


FIG. 1: The brane is embedded as $z = Z(r)$, or equivalently $y = Y(r)$. In this figure, the function $Y(r)$ is shown for $c = 1.5, 2.0, 2.5$ (from up left to down right) with $Z_0 = 0$ and the minus sign in (43).

To summarize, the induced metric on the brane is the metric of 4D asymptotically flat spacetime with a static

black hole in it. We call this object a *brane hole*. The gravitational radius of the brane hole is $r_e = c^2$. And it is located outside the horizon radius of the bulk black string.

There is a simple alternative explanation why the brane-hole horizon is at $r = r_e = c^2$. The 5D mass \mathcal{M} of the black string is

$$\mathcal{M} = ML, \quad (47)$$

where M is a 4D mass (in our case $M = 1/2$) and L is the length of the string (or its segment). Thus the 4D mass can be define as

$$M = \frac{d\mathcal{M}}{dL}. \quad (48)$$

The reference frame in which the black string is at rest is a preferable one. In the reference frame moving with the velocity V the black string energy is $\mathcal{E} = \gamma\mathcal{M}$, where γ is the Lorentz factor, $\gamma = (1 - V^2)^{-1/2}$. At the same time because of the Lorentz contraction the length element in z -direction in the moving frame is $\mathcal{L} = \gamma^{-1}L$. As a result, the 4D energy of the moving black string is

$$\tilde{M} = \gamma^2 M. \quad (49)$$

Now, in our parameterization $c = \gamma$. As a result, the effective gravitational radius, as measured by an observer moving in z -direction, in our units is $r_e = c^2$. This explains the obtained relation (33) for the ‘gravitational radius’ of the brane hole.

V. BRANE HOLES AND THEIR PROPERTIES

The surface gravity of the brane hole depends on the test brane velocity

$$\kappa = \frac{\sqrt{4c^2 - 3}}{2c^3}. \quad (50)$$

When it is not moving, $c = 1$, and the surface gravity κ_0 coincides with the surface gravity of the bulk black string

$$\kappa_0 = 1/2. \quad (51)$$

The plot presented in Figure 2 shows that the ratio $k = \kappa/\kappa_0$, which is equal to 1 at $c = 1$, is greater than one for small non-zero velocity. It reaches the maximum at $c = 3/(2\sqrt{2})$. At

$$c = \frac{\sqrt{\sqrt{13} - 1}}{\sqrt{2}} \quad (52)$$

k takes the value 1 again, and after this it decreases to 0 when $c \rightarrow \infty$.

The brane hole temperature is

$$\Theta = \frac{\kappa}{2\pi} = \frac{\sqrt{4c^2 - 3}}{8\pi M c^3}. \quad (53)$$

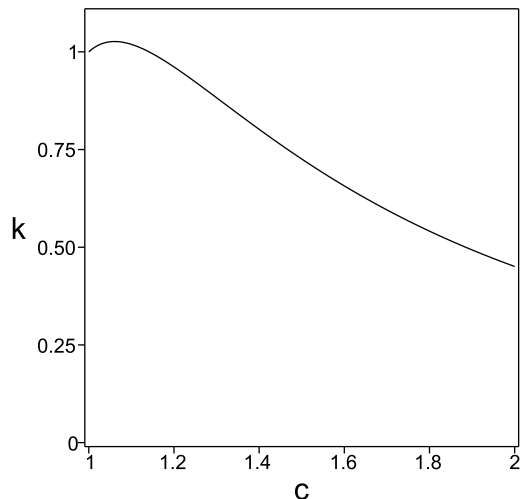


FIG. 2: A plot of the ratio $k = \kappa/\kappa_0$ as a function of c .

We restored the mass M in this formula. For $c = 1$ we obtain the standard expression for the Hawking temperature of a 4D black hole of mass M

$$\Theta_0 = \frac{1}{8\pi M}. \quad (54)$$

As we have seen in the previous section, the horizon radius of the brane hole is $c^2 r_0$, where $r_0 = 2M$ is the horizon radius of the bulk black string. This means that the Misner-Sharp mass of the induced metric at the brane hole horizon is not M but $c^2 M$. On the other hand, the ADM mass, or the Misner-Sharp energy at infinity, is M . Thus, we consider M as energy.

One can easily write the following relation

$$dM = \Theta dS + \mu_c dc. \quad (55)$$

Using this first law we obtain the entropy of the brane hole

$$S = \frac{4\pi c^3}{\sqrt{4c^2 - 3}} M^2 + S_0(c), \quad (56)$$

where $S_0(c)$ is an arbitrary function of c . By demanding that $S = 0$ for $M = 0$, we obtain $S_0(c) = 0$ and thus

$$S = \frac{4\pi c^3}{\sqrt{4c^2 - 3}} M^2, \quad \mu_c = \frac{9 - 8c^2}{2c(4c^2 - 3)} M. \quad (57)$$

For zero velocity case the entropy is

$$S_0 = 4\pi M^2 = \frac{1}{4} \mathcal{A}_0, \quad (58)$$

where \mathcal{A}_0 is the surface area of the 4D Schwarzschild black hole. The surface area of the brane hole is

$$\mathcal{A} = 16\pi c^2 M^2, \quad (59)$$

and one has

$$S = \frac{\beta}{4} \mathcal{A}, \quad \beta = \frac{c}{\sqrt{4c^2 - 3}}. \quad (60)$$

As shown in Figure 3, the ratio β is 1 at $c = 1$ and monotonically decreases towards $1/2$.

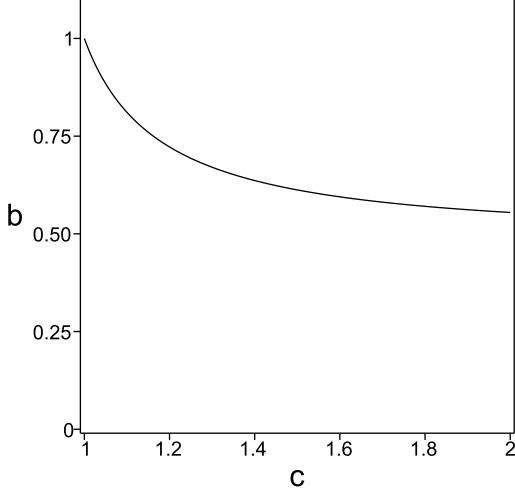


FIG. 3: A plot of β as a function of c .

VI. INFORMATION MINING FROM BRANE-HOLE INTERIOR THROUGH THE EXTRA-DIMENSIONAL ‘WINDOW’

The surface of the test brane besides the induced metric $g_{\mu\nu}$ is also characterized by the extrinsic curvature $K_{\mu\nu}$, which encodes the information about its embedding in the bulk spacetime. The test brane is a minimal surface and its equation (29) is equivalent to the condition

$$Tr K \equiv g^{\mu\nu} K_{\mu\nu} = 0. \quad (61)$$

The calculation of the extrinsic curvature shows that the extrinsic curvature itself does not vanish (see Appendix C), so that the test brane surface is not geodesic. The latter property means that in a generic case a bulk geodesic connecting two point p and p' on the brane differs from a geodesic connecting these points in the induced geometry. We demonstrate now that the motion through the bulk can provide one with a short-cut, so that two points of the brane located at spacelike interval along the brane can be connected by a causal curve in the bulk spacetime.

To illustrate this, let us consider a simple case of a radial null ray in the bulk space. For this problem it is sufficient to work with a 3D metric

$$dS^2 = -\Phi dt^2 + \frac{dr^2}{\Phi} + dz^2. \quad (62)$$

The conserved quantities for the radial $(\dot{t}, \dot{z}, \dot{r})$ motion are

$$\mathcal{E} = \Phi \dot{t}, \quad K = \dot{z}. \quad (63)$$

Here a dot denotes derivative with respect to an affine parameter λ , \mathcal{E} is the energy of the photon, and K is the z -component of its momentum. We choose the affine parameter so that it grows in the ‘future direction’, so that \dot{t} and \mathcal{E} are positive. For the null ray one has $dS^2 = 0$. This relation and (63) give

$$r^2 = \mathcal{E}^2 - K^2 \Phi, \quad \dot{z} = K, \quad \dot{t} = \mathcal{E}/\Phi. \quad (64)$$

It is convenient to exclude the affine parameter λ and parameterize the curve by r . The corresponding equations are

$$\frac{d\tilde{t}}{dr} = \frac{1}{\Phi \sqrt{1 - q^2 \Phi}}, \quad (65)$$

$$\frac{d\tilde{z}}{dr} = \frac{q}{\sqrt{1 - q^2 \Phi}}, \quad (66)$$

where $q = K/\mathcal{E}$. A solution of this set of first order equations determine a null ray trajectory $(\tilde{t}(r), \tilde{z}(r))$. Notice that we chose the sign of the square root, which enters these expression, to be positive. This corresponds to photons propagating outwards.

For points on the moving test brane one has the following relation

$$F \equiv s\tilde{t} - c\tilde{z} + Z(r) = 0. \quad (67)$$

The condition that a null ray meets the moving test brane is

$$\mathcal{F}(r) = s\tilde{t}(r) - c\tilde{z}(r) + Z(r) = 0. \quad (68)$$

Suppose this condition is satisfied at some initial point p where $r = r_1$. In order to determine whether it meets the brane again at larger value of r , it is sufficient to solve the following differential equation

$$\frac{d\mathcal{F}}{dr} = Q, \quad (69)$$

$$Q = \frac{s - cq\Phi}{\Phi \sqrt{1 - q^2 \Phi}} + \frac{sc^3 \sqrt{r}}{(r - 1)\sqrt{U}}, \quad (70)$$

with the initial condition $\mathcal{F}(r_1) = 0$. If a solution passes again through zero at some other radius r_2 , this will determine another point of the intersection of the null ray with the test brane.

Qualitatively it is possible to describe the properties of the function Q as follows. For small radius $r = 1 + \varepsilon$, $Q \sim 2s/\varepsilon$ and is positive. At large radius r it becomes constant and is equal to

$$Q \sim \frac{s - cq}{\sqrt{1 - q^2}}. \quad (71)$$

Hence in this region if $q > s/c = V$, the function Q is negative. This gives us the following asymptotics for the function \mathcal{F}

$$\mathcal{F} \sim 2s\varepsilon(r - 1 - \varepsilon), \quad \text{for } r \approx 1 + \varepsilon, \quad (72)$$

$$\mathcal{F} \sim \mathcal{F}_0 - \frac{qc - s}{\sqrt{1 - q^2}} r. \quad (73)$$

This means that if $q > s/c = V$, the null ray necessarily meets the test brane again at sufficiently large r . The Figure 4 shows the behavior of \mathcal{F} for a special choice of the parameters.

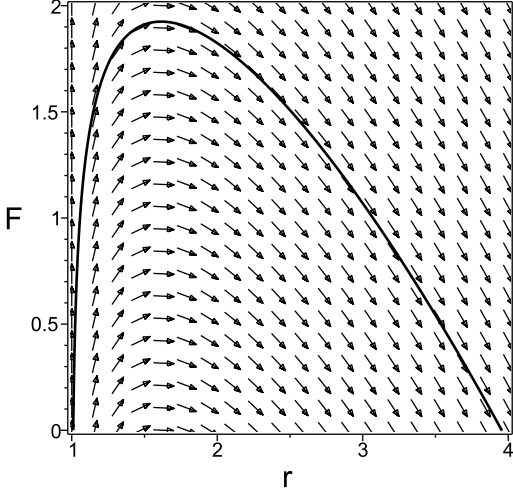


FIG. 4: An example of a bulk null ray connecting brane hole interior with an external region. It shows the function \mathcal{F} with the initial data $\mathcal{F} = 0$ at $r = 1.01$ for following set of parameters: $s = 0.3$, $q = 1.0$. This null ray crosses the test brane again near $r \approx 4.0$.

VII. BRANE HOLES IN THE DARK SHELL GEOMETRY

Let us discuss a model of a test brane which is moving with constant velocity with respect to a dark shell. We again assume that the test brane is asymptotically flat. Near the shell the brane surface it is stretched. We focus on the case in which the radius of the shell $r = 1 + \varepsilon$ is inside the ergosphere radius $r = c^2$. This gives the following condition

$$\varepsilon < s^2. \quad (74)$$

We can 'create' and 'destroy' a brane hole by changing the velocity of the test brane.

Since outside the dark shell the induced metric $g_{+\mu\nu}$ coincides with (36) it possesses a brane hole horizon. Let us emphasize that in such a case a brane hole exists even

when there is no bulk black hole. We study now this interesting case in more detail.

First of all let us calculate the embedding surface of the brane inside the dark shell and obtain the inner induced metric ds_-^2 . The test brane equation is of the form

$$z = Z_-(r). \quad (75)$$

Using expression (21) we find the induced metric

$$ds_-^2 = \frac{1}{1 + \varepsilon} [(s^2 - \varepsilon)dt^2 + 2scZ'_- dt dr + [1 + \varepsilon + (\varepsilon + c^2)(Z'_-)^2]dr^2] + r^2 d\omega^2. \quad (76)$$

The calculation of the determinant of this metric gives

$$\sqrt{-\det g_-} = \frac{\sqrt{\varepsilon(1 + (Z'_-)^2) - s^2}}{\sqrt{1 + \varepsilon}} r^2 \sin^2 \theta. \quad (77)$$

Thus the effective action for the test brane is

$$W_- = \frac{4\pi\Delta T}{\sqrt{1 + \varepsilon}} \int dr L_-, \quad (78)$$

$$L_- = r^2 \sqrt{\varepsilon(1 + (Z'_-)^2) - s^2}. \quad (79)$$

The test brane equation in the inner region

$$\frac{d}{dr} \left(\frac{\partial L_-}{\partial Z'_-} \right) = 0 \quad (80)$$

has a solution

$$Z'_- = \frac{B_-}{\sqrt{\varepsilon}} \sqrt{\frac{s^2 - \varepsilon}{B_-^2 - \varepsilon r^4}}. \quad (81)$$

Here B_- is an arbitrary integration constant. The induced metric for this solution can be made diagonal by means of the transformation

$$t = T - \frac{sc}{s^2 - \varepsilon} \int dr Z'_-. \quad (82)$$

One has

$$ds_-^2 = \frac{s^2 - \varepsilon}{1 + \varepsilon} dT^2 + \frac{1}{\varepsilon - s^2} [\varepsilon(1 + (Z'_-)^2) - s^2] dr^2 + r^2 d\omega^2. \quad (83)$$

The induced metric ds_+^2 taken at the shell surface is

$$ds_+^2 = \frac{s^2 - \varepsilon}{1 + \varepsilon} dT^2 + \frac{(1 + \varepsilon)^4}{(\varepsilon - s^2)U(r = 1 + \varepsilon)} dr^2 + r^2 d\omega^2. \quad (84)$$

Let us notice that the metric coefficient, except g_{rr} , in (83) and (84) are continues on the shell, and $g_{TT} < 0$. From the analysis of the test brane equation it follows that the following condition must be satisfied on the shell (see Appendix D)

$$\frac{Z'_-}{\sqrt{g_{-rr}}} = \frac{Z'_+}{\sqrt{g_{+rr}}}. \quad (85)$$

Solving this equation one finds the value of Z'_- on the shell. Equation (81) at the shell can be used to get $B_- = \pm sc^3\sqrt{1+\varepsilon}$. Substituting B_- back into (81) one finds the function $Z_-(r)$. The result can be written in the form

$$Z'_- = \frac{A}{\sqrt{C^2 - r^4}}, \quad (86)$$

where

$$A = \pm \frac{sc^3}{\varepsilon} \sqrt{(1+\varepsilon)(s^2 - \varepsilon)}, \quad C = |s|c^3 \sqrt{\frac{1+\varepsilon}{\varepsilon}}. \quad (87)$$

Introducing a new variable $x = r/\sqrt{C}$, one obtains

$$Z_- = \frac{A}{\sqrt{C}} \int_0^x \frac{dx}{\sqrt{1-x^4}} = \frac{A}{\sqrt{C}} F(x, i) + Z_{-,0}. \quad (88)$$

Here $F(x, k)$ is the the incomplete elliptic integral of the first kind¹. The constant $Z_{-,0}$ is determined by using the continuity of the function Z at the shell. The function $F(x, i)$ is shown in the Figure 5.

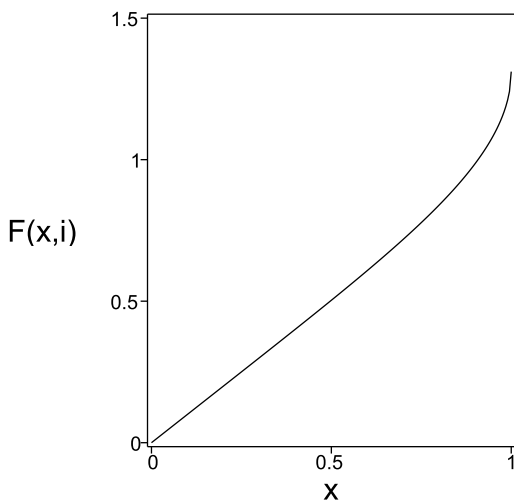


FIG. 5: A plot of the function $F(x, i)$.

The test brane embedding described in this section is singular at the origin $r = 0$. However, this just indicates breakdown of approximations which we have implicitly assumed. Thus it is expected that the apparent singularity at $r = 0$ can be resolved if we take into account effects such as thickness, microscopic physical degrees of freedom of the brane configuration, etc.

VIII. SUMMARY AND DISCUSSIONS

We have studied a *brane hole*, a black hole induced on a test brane in the background of a higher dimensional bulk black string/brane. When the test brane moves with a constant velocity V relative to the bulk black string/brane, the horizon radius of the brane hole r_e is greater than that of the bulk black string/brane r_0 by the factor $(1 - V^2)^{-1}$. We have shown that bulk ‘photons’ emitted in the region between r_0 and r_e can meet the test brane again at a point outside r_e . Therefore, a brane hole provides an explicit example in which extra dimensions can be used to extract information from the interior of a lower dimensional black object.

We have also shown that, even if there is no horizon in the higher dimensional bulk geometry, a moving test brane can still have a brane hole. As a simple example, we have considered a *dark shell* geometry in which a bulk black string/brane is replaced by a massive thin shell located outside the would-be horizon. An interesting feature of this model is that there is no ‘hidden regions’ for the bulk photons, so that a test brane observer interacting with such photons can get information from the complete brane hole interior, including the central region.

In order to realize ordinary 4D gravity at scales longer than $\sim 0.01mm$ in the asymptotic region, the z -direction must be compactified. Also, in the case of a bulk black string/brane, if z -direction were infinite then the system would be dynamically unstable due to Gregory-Laflamme instability [9]. It is well known that compactification of the z -direction suppresses the Gregory-Laflamme instability and can stabilize the bulk black string/brane. For these reasons, we compactify the z -direction by imposing a periodic boundary condition. We can perform this compactification in the rest frame of the test brane. In such a compactified spacetime the picture of the brane hole and its properties remain practically unchanged from what we have described. Main difference which should be mentioned here is that the bulk photons from the brane hole interior can meet the test brane many times.

The concept of brane holes opens up new arenas to investigate black hole evaporation and the information loss problem. Here, let us point out a couple of interesting possibilities.

There are physical degrees of freedom on a test brane such as transverse degrees of freedom of brane fluctuation and matter fields confined on the brane. If we consider them as free fields and simply quantize them on a brane hole background then we would conclude that they exhibit Hawking radiation with the temperature (53). Actually, those degrees of freedom inevitably interact with fields propagating in the bulk such as bulk gravitons. These bulk fields can communicate information stored inside a brane hole to the outside. Thus, if we take into account interactions with bulk fields then the dynamics of quantized brane fields can be quite different. It is certainly worth while seeing if such interactions can help

¹ There are some notational variants for this function. For example, in [8], the same function is denoted as $F(\varphi|\alpha)$ and $F(\varphi|m)$, where $x = \sin \varphi$, $k = \sin \alpha$ and $m = k^2$.

conveying information stored inside the brane hole horizon to the outside.

Another interesting issue would be creation and annihilation of a brane hole. In the dark shell model, there is a critical velocity V_c below which the test brane does not contain a brane hole but above which a brane hole appears on the test brane. Therefore, by controlling the test brane velocity relative to the rest frame of the dark shell, one can create and annihilate a brane hole. Possible ways to control the test brane velocity are accretion and superradiance of Kaluza-Klein particles. It is interesting to investigate them in details.

Acknowledgments

This work was initiated during V.F.'s visit to IPMU. He is thankful to IPMU for their kind hospitality. Part of this work was done during APCTP Joint Focus Program: Frontiers of Black Hole Physics. The authors thank APCTP for stimulating atmosphere and warm hospitality. One of the authors (V.F.) thanks the Natural Sciences and Engineering Research Council of Canada and the Killam Trust for their support. The work of S.M. is supported by Grant-in-Aid for Scientific Research 17740134, 19GS0219, 21111006, 21540278, by World Premier International Research Center Initiative (WPI Initiative), and by the Mitsubishi Foundation.

Appendix A: Higher dimensional generalizations

Our starting point is the metric $(a, b = 0, 1, \dots, D-1)$

$$dS^2 = g_{ab}dy^a dy^b = -\Phi d\tilde{t}^2 + \frac{dr^2}{H} + r^2 d\omega^2 + d\tilde{z}^2. \quad (\text{A1})$$

The total number of the spacetime dimensions is $D = 4 + m$,

$$d\tilde{z}^2 = d\tilde{z}_1^2 + \sum_{i=2}^m dz_i^2 \quad (\text{A2})$$

is the flat m -dimensional metric, $d\omega^2$ is a metric of a unit round 2-sphere S^2 . (It is straightforward to consider an n -sphere S^n ($n \geq 3$) instead of a 2-sphere S^2 . However, in this case the number of non-compact dimensions would become more than 4. Thus, we shall concentrate on the case with S^2 .) For

$$\Phi = H = 1 - \varphi, \quad \varphi = r_0/r, \quad (\text{A3})$$

the metric (A1) is a solution of the vacuum Einstein equation. This solution is a direct sum of $4D$ Schwarzschild metric and the m -dimensional flat metric. It is called a *black brane*. Since we have already considered the case with $m = 1$, in this appendix we shall suppose that $m \geq 2$.

A boosted black brane can be obtained by the transformation

$$\tilde{t} = ct + sz_1, \quad \tilde{z}_1 = st + cz_1, \quad (\text{A4})$$

$$c = \cosh \alpha, \quad s = \sinh \alpha. \quad (\text{A5})$$

Let us consider a $(\bar{m} + 3)$ -brane ($0 \leq \bar{m} \leq m - 1$) whose worldvolume is specified by the embedding

$$z_1 = Z(r), \quad z_j = z_{j,0} \quad (j = \bar{m} + 2, \dots, m), \quad (\text{A6})$$

where $z_{j,0}$ are constants. This brane fills not only the $4D$ Schwarzschild geometry but also \bar{m} -dimensional flat extra dimensions $(z_2, \dots, z_{\bar{m}+1})$. One can compactify all z_i ($i = 1, \dots, m$) on a m -torus.

The rest of the calculations are essentially the same as those presented in the main text and, thus, we shall not repeat them here.

Appendix B: Ricci tensor for the induced metric

The Ricci tensor for the metric (36) is

$$R_\mu^\nu = \text{diag}(R_0, R_1, R_2, R_3), \quad (\text{B1})$$

$$R_0 = 1/4r^{-6}c^2(3c^6 - 3c^4 + 2c^4r + c^2r^2 - 2c^2r - r^2), \quad (\text{B2})$$

$$R_1 = -1/4r^{-6}(9c^8 - 9c^6 - 6rc^6 - 5c^4r^2 + 6c^4r + 5c^2r^2 - 4c^2r^3 + 4r^3), \quad (\text{B3})$$

$$R_2 = -1/2r^{-6}(3c^8 - 3c^6 + rc^6 - c^4r + c^4r^2 - c^2r^2 + c^2r^3 - r^3), \quad (\text{B4})$$

$$R_3 = -1/2r^{-6}(3c^8 - 3c^6 + rc^6 - c^4r + c^4r^2 - c^2r^2 + c^2r^3 - r^3). \quad (\text{B5})$$

Appendix C: Extrinsic curvature

In the absence of the boost, when $c = 1$, the solution for the brane is $z = 0$. The induced metric coincides with the $4D$ Schwarzschild metric. The radius of the brane hole is r_0 . The embedding equation is symmetric with respect to reflection $z \rightarrow -z$. As a result, the brane surface is geodesically embedded. This property is not true any more when one has a non-vanishing boost. To see this explicitly let us calculate the extrinsic curvature.

Using the coordinates (t, r, θ, ϕ, z) in the bulk space, the $5D$ metric is

$$dS^2 = -(1 - c^2/r)dt^2 + \frac{2sc}{r}dtdz + \frac{dr^2}{1 - 1/r} + r^2d\omega^2 + (1 + s^2/r)dz^2, \quad (\text{C1})$$

and the equation of the brane Σ is $F = z - Z(r) = 0$. A unit normal vector to the surface Σ is

$$n_\mu = (0, -\frac{sc^3}{r(r-1)}, \sqrt{\frac{U}{r^3}}, 0, 0). \quad (\text{C2})$$

Here U is defined by (37).

Let us introduce the following vectors

$$\begin{aligned} e_0^a &= \left(\sqrt{\frac{r}{r-c^2}}, 0, 0, 0, 0 \right), \\ e_1^a &= \left(\frac{s^2 c^4}{\sqrt{r-c^2} r^{3/2} (r-1)}, \frac{\sqrt{(r-c^2)U}}{r^2}, 0, 0, \frac{c^3 s \sqrt{r-c^2}}{r^{3/2} (r-1)} \right), \\ e_2^a &= (0, 0, r^{-1}, 0, 0), \\ e_3^a &= (0, 0, 0, (r \sin \theta)^{-1}, 0). \end{aligned} \quad (C3)$$

The vectors e_μ^a are tangent to Σ , e_0^a being a timelike. The vectors e_μ^a and n^a are mutually orthogonal and have a unit norm.

The extrinsic curvature is

$$K_{\hat{\mu}\hat{\nu}} = e_{\hat{\mu}}^a e_{\hat{\nu}}^b n_{a|b}. \quad (C5)$$

The non-vanishing components of the extrinsic curvature are

$$K_{\hat{0}\hat{0}} = \frac{sc^5}{2r^3(r-c^2)}, \quad (C6)$$

$$K_{\hat{1}\hat{1}} = \frac{sc^3(4r-3c)}{2r^3(r-c^2)}, \quad (C7)$$

$$K_{\hat{0}\hat{1}} = \frac{sc}{2r(r-c^2)}, \quad (C8)$$

$$K_{\hat{2}\hat{2}} = K_{\hat{3}\hat{3}} = -\frac{sc^3}{r^3}. \quad (C9)$$

Direct check shows that

$$\text{Tr}K = -K_{\hat{0}\hat{0}} + K_{\hat{1}\hat{1}} + K_{\hat{2}\hat{2}} + K_{\hat{3}\hat{3}} = 0. \quad (C10)$$

The relation $\text{Tr}K = 0$ must be valid, since the surface Σ is a minimal surface. Thus (C10) is simply a consistency check of the calculation.

Appendix D: Junction across dark shell

We consider a 5-dimensional geometry without horizon

$$dS^2 = -dt^2 + dz^2 + \tilde{\varphi}(r)(cdt + sdz)^2 + \frac{dr^2}{H(r)} + r^2 d\omega^2, \quad (D1)$$

where $c = \cosh \alpha$, $s = \sinh \alpha$ and α is a boost parameter. For a brane embedding of the form

$$z = Z(r), \quad (D2)$$

the induced metric is

$$ds^2 = -(1-c^2\tilde{\varphi})dT^2 + \left[\frac{1-\tilde{\varphi}}{1-c^2\tilde{\varphi}}(Z')^2 + \frac{1}{H} \right] dr^2 + r^2 d\omega^2, \quad (D3)$$

where

$$T = t - sc \int \frac{dr \tilde{\varphi} Z'}{1-c^2\tilde{\varphi}}. \quad (D4)$$

Thus the brane action is

$$W = \int dT \int_0^{2\pi} d\phi \int_0^\pi d\theta \int dr \sqrt{-g} = 4\pi \Delta T \int dr L, \quad (D5)$$

where

$$L = r^2 \sqrt{\frac{(1-\tilde{\varphi})H(Z')^2 + 1 - c^2\tilde{\varphi}}{H}}. \quad (D6)$$

The equation of motion is

$$B' = 0, \quad B \equiv \frac{r^2(1-\tilde{\varphi})H^{1/2}Z'}{[(1-\tilde{\varphi})H(Z')^2 + 1 - c^2\tilde{\varphi}]^{1/2}}. \quad (D7)$$

Thus, B is constant. The definition of B can be solved with respect to Z' as

$$Z' = B \sqrt{\frac{1 - c^2\tilde{\varphi}}{(1-\tilde{\varphi})H[r^4(1-\tilde{\varphi}) - B^2]}} \quad (D8)$$

Let us now introduce two small parameters ε and $\tilde{\varepsilon}$ so that

$$0 < \tilde{\varepsilon} \ll \varepsilon < 1, \quad (D9)$$

and smoothly connect the flat spacetime to a curved spacetime as

$$\begin{aligned} \tilde{\varphi}(r) &= \begin{cases} \frac{1}{1+\varepsilon} & (r < 1 + \varepsilon - \tilde{\varepsilon}/2) \\ \tilde{\varphi}_*(r) & (1 + \varepsilon - \tilde{\varepsilon}/2 \leq r \leq 1 + \varepsilon + \tilde{\varepsilon}/2) \\ \varphi(r) & (r > 1 + \varepsilon + \tilde{\varepsilon}/2) \end{cases}, \\ H(r) &= \begin{cases} 1 & (r < 1 + \varepsilon - \tilde{\varepsilon}/2) \\ H_*(r) & (1 + \varepsilon - \tilde{\varepsilon}/2 \leq r \leq 1 + \varepsilon + \tilde{\varepsilon}/2) \\ 1 - \varphi(r) & (r > 1 + \varepsilon + \tilde{\varepsilon}/2) \end{cases}, \end{aligned} \quad (D10)$$

where $\varphi(r) = 1/r$ and the subscript “*” represents functions smoothly connecting the inside region and the outside region.

For $s^2 > \varepsilon$, the induced metric on the brane has a horizon. The regularity of the horizon determines the constant B as

$$B = \pm sc^3. \quad (D11)$$

Thus, we obtain

$$Z' = \begin{cases} \pm sc^3 \sqrt{\frac{(1+\varepsilon)(s^2-\varepsilon)}{\varepsilon[(1+\varepsilon)s^2c^6-\varepsilon r^4]}} & (r < 1 + \varepsilon - \tilde{\varepsilon}/2) \\ \pm \frac{sc^3 \sqrt{r}}{(r-1)\sqrt{U}} & (r > 1 + \varepsilon + \tilde{\varepsilon}/2) \end{cases}, \quad (D12)$$

where

$$U = r^3 + s^2(r^2 + c^2 r + c^4). \quad (D13)$$

Finally, we can safely take the limit $\tilde{\varepsilon} \rightarrow 0$. We conclude that $\sqrt{H}Z'$ is continuous in this limit.

-
- [1] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **429**, 263 (1998) [arXiv:hep-ph/9803315].
 - [2] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **436**, 257 (1998) [arXiv:hep-ph/9804398].
 - [3] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Rev. D **59**, 086004 (1999) [arXiv:hep-ph/9807344].
 - [4] V. P. Frolov, S. Hendy and A. L. Larsen, Phys. Rev. D **54**, 5093 (1996) [arXiv:hep-th/9510231].
 - [5] V. P. Frolov and A. L. Larsen, Nucl. Phys. B **449**, 149 (1995) [arXiv:hep-th/9503060].
 - [6] V. P. Frolov and D. Fursaev, Phys. Rev. D **63**, 124010 (2001) [arXiv:hep-th/0012260].
 - [7] W. Israel, Nuovo Cim. B **44S10**, 1 (1966) [Erratum-ibid. B **48**, 463 (1967)] [Nuovo Cim. B **44**, 1 (1966)].
 - [8] M. Abramowitz and I. A. Stegun (ed.), *Handbook of Mathematical Functions* (Dover, New York, 1972).
 - [9] R. Gregory and R. Laflamme, Phys. Rev. Lett. **70**, 2837 (1993) [arXiv:hep-th/9301052].